J Glob Optim (2011) 49:23–35
DOI 10.1007/s10898-009-9522-z

Optimality and duality in vector optimization involving generalized type I functions over cones

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Received: 12 January 2008 / Accepted: 24 December 2009 / Published online: 16 January 2010

Abstract In this paper generalized type-I, generalized quasi type-I, generalized pseudo type-I and other related functions over cones are defined for a vector minimization problem. Sufficient optimality conditions are studied for this problem using Clarke's generalized gradients. A Mond-Weir type dual is formulated and weak and strong duality results are established.

Keywords Vector optimization · Cones · Invexity · Type-I functions · Optimality · Duality

1 Introduction

Convexity plays a key role in optimality and duality of mathematical programming problems. Various attempts have been made during the past several decades to weaken convexity hypothesis [2,11,12]. In this endeavor, Hanson [6] defined a new class of functions with applications to optimization theory, which was called the class of invex functions by Craven [5]. Kaul and Kaur [8] called these differentiable invex functions as η-convex and studied their generalizations namely η-pseudoconvex and η-quasiconvex functions. In a very recent work, Antzak [1] considered η-approximation method for solving a nonlinear constrained mathematical programming problem involving invex functions. Khurana [9] introduced differentiable cone-pseudoinvex and strongly cone-pseudoinvex functions as a generalization of pseudoinvex functions.

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Based on the work of Craven [4], Reiland [13] extended the concept of invexity to nonsmooth Lipschitz functions. Yen and Sach [19] defined cone-generalized invex and cone-nonsmooth invex functions. Suneja et al. [17] introduced the concepts of cone-nonsmooth quasi invex, cone-nonsmooth pseudo invex and other related functions in terms of Clarke's [3] generalized directional derivatives and used them to obtain optimality and duality results for a nonsmooth vector optimization problem.

Hanson and Mond [7] introduced two classes of functions called type-I and type-II as generalizations of invex functions. Rueda and Hanson [14] defined pseudo type-I and quasi type-I functions and obtained sufficient optimality conditions for a nonlinear programming problem. Zhao [20] gave Karush-Kuhn Tucker type sufficiency conditions and duality results in non-differentiable scalar optimization problems under type-I functions.

Suneja et al. [18] investigated optimality and duality results under generalized type-I assumptions for nonsmooth multiobjective fractional programming problems. Kuk and Tanino [10] studied optimality and duality results for a nonsmooth multiobjective optimization problem involving generalized type–I functions.

In this paper, we introduce generalized type-I, generalized quasi type-I, generalized pseudo type-I, generalized quasi pseudo type-I and generalized pseudo quasi type-I functions over cones, for a nonsmooth vector optimization problem using Clarke's generalized gradients of locally Lipschitz functions. Sufficient optimality conditions are established and a Mond-Weir type dual is associated with the optimization problem. Weak and strong duality results are proved for the pair under cone generalized type-I assumptions.

2 Preliminaries and definitions

Let $K \subseteq R^m$ be a closed convex cone with non empty interior and int K and $\bar{K}$ denote the interior and closure of cone K respectively. The positive dual cone K^+ of K is defined as

$$K^+ = \{y^* \in R^m : \langle y, y^* \rangle \geq 0, \quad \text{for all } y \in K\}.$$

The strict positive dual cone K^{S+} is given by

$$K^{S+} = \left\{y^* \in R^m : \langle y, y^* > 0 \rangle, \quad \text{for all } y \in K\backslash \{0\}\right\}.$$

A real valued function φ: $R^n \to R$ is said to be locally Lipschitz at a point $u \in R^n$ if there exists a number $\ell > 0$ such that

$$||\varphi(x) - \varphi(\bar{x})|| \leq \ell ||x - \bar{x}||,$$

for all x, $\bar{x}$ in a neighbourhood of u. A function is said to be locally Lipschitz on R^n, if it is locally Lipschitz at each point of R^n.

Definition 1 [3] Let φ: $R^n \to R$ be locally Lipschitz, then $\varphi^\circ(u; v)$ denotes the Clarke's generalized directional derivative of φ at $u \in R^n$ in the direction v and is defined as

$$\varphi^o(u; v) = \limsup_{\substack{y \to u \\ t \to 0}} \frac{\varphi(y + tv) - \varphi(y)}{t}.$$

Clarke's generalized gradient of φ at u is denoted by $\partial\varphi(u)$ and is defined as

$$\partial\varphi\,(u) = \{\xi \in R^n : \varphi^\circ\,(u; v) \geq \langle \xi, v \rangle, \quad \text{for all } v \in R^n\}.$$

Let $f : R^n \to R^m$ be a vector valued function given by $f = (f_1, f_2, \ldots, f_m)$, $f_i : R^n \to R$. Then f is said to be locally Lipschitz on R^n if each f_i is locally Lipschitz on R^n. The generalized directional derivative of a locally Lipschitz function $f : R^n \to R^m$ at $u \in R^n$ in the direction v is given by

$$f^0(u; v) = \{f_1^0(u; v), f_2^0(u; v), \ldots, f_m^0(u; v)\}.$$

The generalized gradient of f at u is the set

$$\partial f(u) = \partial f_1(u) \times \cdots \times \partial f_m(u),$$

where $\partial f_i(u)$ is the generalized gradient of f_i at u, $i = 1, 2, \ldots, m$.

Every element $A = (r_1, r_2, \ldots, r_m) \in \partial f(u)$ is a continuous linear operator from R^n to R^m and

$$Au = (\langle r_1, u\rangle, \ldots, \langle r_m, u\rangle) \in R^m, \quad \text{for all } u \in R^n.$$

The following properties of Clarke's generalized gradient shall be used in the paper.

Lemma 1 *(i) If* $f_i : R^n \to R$ *is locally Lipschitz then, for each* $u \in R^n$,

$$f_i^0(u; v) = \max\{\langle \xi, v\rangle : \xi \in \partial f_i(u)\}.$$

(ii) Let f_i *(*$i = 1, 2, \ldots, n$*) be a finite family of locally Lipschitz functions on* R^n*, then* $\sum_{i=1}^{n} f_i$ *is also locally Lipschitz and*

$$\partial\left(\sum_{i=1}^{n} f_i\right)(u) \subseteq \sum_{i=1}^{n} \partial f_i(u), \quad \text{for every } u \in R^n.$$

We consider the vector minimization problem

$$\text{(VP)} \quad K\text{-}minimize\ f(x)$$
$$subject\ to\ -g(x) \in Q.$$

where $f : R^n \to R^m$, $g : R^n \to R^p$ *are locally Lipschitz functions on* R^n*, K and Q are closed convex cones with nonempty interiors in* R^m *and* R^p *respectively.*

Let $X_0 = \{x \in R^n : -g(x) \in Q\}$ *be the feasible set of (VP).*

For the vector optimization problem (VP), the solutions are defined in the following sense.

Definition 2 Let $\bar{x} \in X_0$, then

(i) $\bar{x}$ is called a weak minimum of (VP) if for all $x \in X_0$, $f(\bar{x}) - f(x) \notin \text{int}\, K$.
(ii) $\bar{x}$ is a minimum of (VP) if for all $x \in X_0$, $f(\bar{x}) - f(x) \notin K/\{0\}$.
(iii) $\bar{x}$ is a strong minimum of (VP) if for all $x \in X_0$, $f(x) - f(\bar{x}) \in K$.
(iv) $\bar{x}$ is called a Benson proper minimum of (VP) if

$$(-K) \cap \text{clcone}\,(f(X_0) + K - f(\bar{x})) = \{0\}$$

Yen and Sach [19] defined K-generalized invex and K-nonsmooth invex functions as given below:

Let $f : R^n \to R^m$ be a locally Lipschitz function on R^n.

Definition 3 [17,19] f is said to be K-generalized invex at the point $u \in R^n$ if there exists $\eta : R^n \times R^n \to R^n$ such that for every $x \in R^n$, $A \in \partial f(u)$

$$f(x) - f(u) - A\eta(x, u) \in K.$$

Definition 4 [17,19] f is said to be K-nonsmooth invex at $u \in R^n$, if there exists $\eta : R^n \times R^n \to R^n$ such that for every $x \in R^n$, $f(x) - f(u) - f^\circ(u; \eta) \in K$.

The above defined functions are respectively called K-invex and K-invex in the limit by Yen and Sach [19].

We now introduce generalized type-I and nonsmooth type-I functions over cones for the problem (VP):

Definition 5 (f, g) is said to be (K × Q) generalized type-I at the point $u \in R^n$, if there exists η: $X_0 \times R^n \to R^n$ such that for every $x \in X_0$ and $A \in \partial f(u)$, $B \in \partial g(u)$,

$$f(x) - f(u) - A\eta(x, u) \in K \tag{1}$$

$$-g(u) - B\eta(x, u) \in Q. \tag{2}$$

Definition 6 (f, g) is said to be (K × Q) nonsmooth type-I at $u \in R^n$, if there exists $\eta : X_0 \times R^n \to R^n$ such that for every $x \in X_0$

$$f(x) - f(u) - f^0(u; \eta) \in K$$

$$-g(u) - g^0(u; \eta) \in Q.$$

Remark 1 If $f : R^n \to R$ and $g : R^n \to R^m$, $K = R_+$ and $Q = R_+^m$ then the above definition reduces to type I invex functions defined by Sach et al. [15].

Lemma 2 *If* (f, g) *is* (K × Q) *generalized type-I at* $u \in R^n$ *with respect to* $\eta : X_0 \times R^n \to R^n$ *then* (f, g) *is* (K × Q)*-nonsmooth type-I with respect to same* η*.*

Proof Let (f, g) be (K × Q) generalized type-I at $u \in R^n$, with respect to $\eta : X_0 \times R^n \to R^n$ then for every $x \in R^n$, for all $A \in \partial f(u)$, $B \in \partial g(u)$,

$$f(x) - f(u) - A\eta(x, u) \in K$$

and $-g(u) - B\eta(x, u) \in Q$.

For each index $i \in \{1, 2, \ldots, m\}$, choose $\bar{v}_i \in \partial f_i(u)$ such that

$$\langle \bar{v}_i, \eta\rangle = \sup\{\langle v_i, \eta\rangle : v_i \in \partial f_i(u)\} = f_i^0(u; \eta)$$

then $\bar{A} = (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m) \in \partial f(u)$ and $f(x) - f(u) - \bar{A}\eta(x; u) \in K$ which implies that,

$$f(x) - f(u) - f^0(u; \eta) \in K.$$

Similarly, for each $j \in \{1, 2, \ldots, p\}$ we can choose $\bar{w}_j \in \partial g_j(u)$ and obtain $\bar{B} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_p) \in \partial g(u)$.

Thus $-g(u) - \bar{B}\eta(x, u) \in Q$,
which gives that

$$-g(u) - g^\circ(u; \eta) \in Q.$$

Hence (f, g) is (K × Q) non smooth type-I at u, with respect to same η.

The following example shows that the converse of the above lemma is not true. □

Example 1 Consider the problem (VP) where $n = 1, m = 2, p = 2$,

$$K = \{(x, y) : x \le y\} \quad \text{and} \quad Q = \{(x, y) : x \ge y\}.$$

Let

$$f_1(x) = \begin{cases} x-1, & x<0 \\ -1, & x \geq 0 \end{cases} \qquad f_2(x) = \begin{cases} x-x^3, & x<0 \\ 0, & x \geq 0 \end{cases}$$

$$g_1(x) = \begin{cases} 1+x, & x<0 \\ 1, & x \geq 0 \end{cases} \qquad g_2(x) = \begin{cases} 2, & x<0 \\ 2-x^2, & x \geq 0 \end{cases}$$

then $-g(x) \in Q \Rightarrow x \leq 1$

Therefore the feasible set becomes $X_0 = \{x \in R: x \leq 1\}$.

Define $\eta : X_0 \times R \to R$ as $\eta(x, u) = (x-u)^3$.

Now, $\partial f_1(0) = [0, 1]$, $f_1^0(0, v) = \begin{cases} 0, & v<0 \\ v, & v \geq 0 \end{cases}$

$$\partial f_2(0) = [0, 1], \quad f_2^0(0, v) = \begin{cases} 0, & v<0 \\ v, & v \geq 0 \end{cases}$$

therefore

$$f(x) - f(0) - f^0(0; \eta) \in K.$$

Further $-g(0) - g^0(0; \eta) \in Q$, for every $x \in X_0$.

Hence (f, g) is $(K \times Q)$-nonsmooth type I at $u = 0$.

However, for $A_1 = \frac{1}{4} \in \partial f_1(0)$, $\quad A_2 = \frac{1}{2} \in \partial f_2(0)$ and $x = 1 \in X_0$, $A = (A_1, A_2)$

$$f(x) - f(0) - A\eta(x, 0) = \left(\frac{-1}{4}, \frac{-1}{2}\right) \notin K.$$

Thus, (f, g) is not $(K \times Q)$ generalized type-I at $u = 0$, with respect to η defined above.

Remark 2 If f is K-generalized invex and g is Q-generalized invex at $u \in R^n$ with respect to same $\eta : X_0 \times R^n \to R^n$, then (f, g) is $(K \times Q)$ generalized type-I at u. However the converse is not true as can be seen from the following example.

Example 2 Consider the problem

$$\begin{aligned} &\text{K-minimize } f(x) \\ &\text{subject to } \quad -g(x) \in Q, \end{aligned}$$

$f : R \to R^2$, $g : R \to R^2$, $f(x) = (f_1(x), f_2(x))$, $g(x) = (g_1(x), g_2(x))$,

$K = \{(x, y): x \leq 0, y \leq -x\}$ and $Q = \{(x, y) : x \leq y, y \geq 0\}$.

Let

$$f_1(x) = \begin{cases} x, & x<0 \\ -x^2, & x \geq 0 \end{cases} \qquad f_2(x) = \begin{cases} 1+x, & x<0 \\ 1, & x \geq 0 \end{cases}$$

$$g_1(x) = \begin{cases} -x^2+3, & x<0 \\ x+3, & x \geq 0 \end{cases} \qquad g_2(x) = \begin{cases} 0, & x<0 \\ -\frac{x}{2} & x \geq 0 \end{cases}$$

then, $-g(x) \in Q \Rightarrow x \geq -\sqrt{3}$. Hence $X_0 = \{ x \in R: \geq -\sqrt{3}\}$

Here $\partial f_1(0) = [0, 1]$, $\partial f_2(0) = [0, 1]$, $\partial g_1(0) = [0, 1]$, $\partial g_2(0) = [-1/2, 0]$

Define $\eta : X_0 \times R \to R$ as $\eta(x, u) = (x-u)^2$.

Then, (f, g) is $(K \times Q)$ generalized type-I at $u = 0$, with respect to η defined above.

As $\quad f(x) - f(0) - A\eta(x, 0) \in K$, for every $x \in X_0$, $A \in \partial f(0)$

and $\quad -g(0) - B\eta(x, 0) \in Q$, for every $x \in X_0$ and $B \in \partial g(0)$.

Further, for $B_1 = \frac{1}{2} \in \partial g_1(0), B_2 = \frac{-1}{4} \in \partial g_2(0)$ and $x = 1 \in X_0$,

$$g(x) - g(0) - B\eta(x, 0) = \left(\frac{1}{2}, \frac{-1}{4}\right) \notin Q.$$

Therefore (f, g) is not (K × Q) generalized invex at u = 0.

Remark 3 If $K = R^m_+, Q = R^p_+$ and (f, g) is (K × Q) generalized type-I with respect to η then (f, g) is type-I [10] with respect to same η. However, the converse fails as can be viewed from the following example.

Example 3 The pair (f, g) considered in Example 2 has been proved to be (K × Q) generalized type-I with respect to $\eta(x, u) = (x - u)^2$. However (f, g) is not generalized type I with respect to same η, because, for $x = 1 \in X_0$, $A_1 = \frac{1}{2}$, $f_1(x) - f_1(0) - A_1\eta(x, 0) = \frac{-3}{2} < 0$ and for $B_1 = \frac{1}{4}$,

$$-g_1(0) - B_1\eta(x, 0) = \frac{-13}{4} < 0.$$

Based on the lines of Soleimani-damaneh [16] we now introduce various generalizations of (K × Q) generalized type-I functions.

Definition 7 (f, g) is said to be (K × Q) generalized pseudo type-I at $u \in R^n$ if there exists η: $X_0 \times R^n \to R^n$ such that for all $x \in X_0$, some $A \in \partial f(u)$, $B \in \partial g(u)$,

$$-A\eta(x, u) \notin \text{int } K \Rightarrow -(f(x) - f(u)) \notin \text{int } K.$$
$$-B\eta(x, u) \notin \text{int } Q \Rightarrow g(u) \notin \text{int } Q.$$

In other words,
(f, g) is said to be (K × Q) generalized pseudo-type-I at $u \in R^n$, if there exists $\eta : X_0 \times R^n \to R^n$ such that for every $x \in X_0$ and for all $A \in \partial f(u)$, $B \in \partial g(u)$,

$$f(u) - f(x) \in \text{int } K \Rightarrow -A\eta(x, u) \in \text{int } K.$$
$$g(u) \in \text{int } Q \Rightarrow -B\eta(x, u) \in \text{int } Q.$$

Definition 8 (f, g) is said to be (K × Q) generalized quasi type-I at $u \in R^n$, if there exists $\eta : X_0 \times R^n \to R^n$ such that for every $x \in X_0$, for all $A \in \partial f(u)$, $B \in \partial g(u)$

$$f(x) - f(u) \notin \text{int } K \Rightarrow -A\eta(x, u) \in K$$
$$-g(u) \notin \text{int } Q \Rightarrow -B\eta(x, u) \in Q.$$

Definition 9 (f, g) is said to be (K × Q) generalized pseudo quasi type-I at $u \in R^n$, if there exists $\eta : X_0 \times R^n \to R^n$ such that for every $x \in X_0$, for all $A \in \partial f(u)$, $B \in \partial g(u)$

$$f(u) - f(x) \in \text{int } K \Rightarrow -A\eta(x, u) \in \text{int } K$$
$$-g(u) \notin \text{int } Q \Rightarrow -B\eta(x, u) \in \text{int } Q.$$

Definition 10 (f, g) is said to be (K × Q) generalized quasi pseudo type-I at $u \in R^n$, if there exists $\eta : X_0 \times R^n \to R^n$ such that for every $x \in X_0$ for all $A \in \partial f(u)$

$$f(x) - f(u) \notin \text{int } K \Rightarrow -A\eta(x, u) \in K$$
$$-B\eta(x, u) \notin \text{int } Q \Rightarrow g(u) \notin \text{int } Q, \text{ for some } B \in \partial g(u).$$

Definition 11 (f, g) is said to be strictly (K × Q) generalized pseudo quasi type-I at $u \in R^n$, if there exists $\eta : X_0 \times R^n \to R^n$, such that for every $x \in X_0$, for all $A \in \partial f(u)$, $B \in \partial g(u)$

$$f(u) - f(x) \in K \Rightarrow -A\eta(x, u) \in \text{int } K$$
$$-g(u) \notin \text{int } Q \Rightarrow -B\eta(x, u) \in Q.$$

Definition 12 (f, g) is said to be strongly (K × Q) generalized pseudo quasi type-I at $u \in R^n$, if there exists $\eta : X_0 \times R^n \to R^n$ such that for every $x \in X_0$,

$$-A\eta(x, u) \notin \text{int } K \Rightarrow f(x) - f(u) \in K, \text{ for some } A \in \partial f(u)$$
$$-g(u) \notin \text{int } Q \Rightarrow -B\eta(x, u) \in Q, \text{ for all } B \in \partial g(u).$$

We now give an example of a function which is (K × Q)-generalized pseudo type-I but not (K × Q) generalized type-I.

Example 4 Consider the problem

$$\text{K-minimize } f(x)$$
$$\text{subject to } -g(x) \in Q,$$

where $f : R \to R^2$, $g : R \to R^2$, $f(x) = (f_1(x), f_2(x))$, $g(x) = (g_1(x), g_2(x))$

$$K = \{(x, y) : y \le x, x \ge 0\}, Q = \{(x, y) : y \le -x\}$$
$$f_1(x) = \begin{cases} -x^2, & x < 1 \\ -x, & x \ge 1 \end{cases} \quad f_2(x) = \begin{cases} x, & x < 1 \\ x^3, & x \ge 1 \end{cases}$$
$$g_1(x) = \begin{cases} x, & x < 1 \\ 1, & x \ge 1 \end{cases} \quad g_2(x) = \begin{cases} 0 & x < 1 \\ -x + 1, & x \ge 1 \end{cases}$$

Now $-g(x) \in Q \Rightarrow 0 \le x \le 2$, therefore the feasible set is

$$X_0 = \{x \in R : 0 \le x \le 2\}.$$

Here $\partial f_1(1) = [-1, 0]$, $\partial f_2(1) = [0, 1]$, $\partial g_1(1) = [0, 1]$ and $\partial g_2(1) = [-1, 0]$.

Define $\eta : X_0 \times R \to R$ as $\eta(x, u) = x^2 - u^2$

Then (f, g) is (K × Q) generalized pseudo type-I at $u = 1$, with respect to η because for $A = (-1, 1) \in \partial f(1)$

$$-A\eta(x, 1) \notin \text{int } K \Rightarrow -(f(x) - f(1)) \notin \text{int } K$$

and for $B = (1, -1) \in \partial g(1)$, $-B\eta(x, 1) \notin \text{int } Q \Rightarrow g(1) \notin \text{int } Q$.

On the other hand,

for $A = (-1, 0)$ and $x = \frac{3}{2} \in X_0$,

$$f(x) - f(1) - A\eta(x,1) = \left(\frac{3}{4}, \frac{19}{8}\right) \notin K.$$

Hence (f, g) fails to be (K × Q) generalized type-I.

We now give an example to show that (K × Q) generalized pseudo quasi type-I function may fail to be (K × Q) generalized type-I.

Example 5 Consider the problem

$$\begin{aligned}&\text{K-minimize } f(x)\\ &\text{subject to } -g(x)\in Q,\end{aligned}$$

where $K=\{(x,y): x\geq 0, y\leq x\}$, $Q=\{(x,y): x\leq 0, y\leq -x\}$.

$f: R\to R^2$, $f=(f_1,f_2)$; $g: R\to R^2$, $g=(g_1,g_2)$

$$f_1(x) = \begin{cases}1, & x<0\\ -x+1, & x\geq 0\end{cases} \quad f_2(x)=\begin{cases}x-2 & x<0\\ -2 & x\geq 0\end{cases}$$

$$g_1(x) = \begin{cases}x, & x<0\\ x^2, & x\geq 0\end{cases} \quad g_2(x)=\begin{cases}0, & x<0\\ x^3-x^2 & x\geq 0\end{cases}$$

Then, $-g(x)\in Q \Rightarrow x\geq 0$, therefore the feasible set is $X_0=\{x\in R: x\geq 0\}$.

Here $\partial f_1(0)=[-1,0]$, $\partial f_2(0)=[0,1]$, $\partial g_1(0)=[0,1]$, $\partial g_2(0)=\{0\}$

Define $\eta: X_0\times R\to R$ as $\eta(x,u)=x^3-u^3$.

Then (f, g) is $(K\times Q)$ generalized pseudo quasi type-I at $u=0$ with respect to η because, for $A=(-1,1)\in\partial f(0)$

$$-A\eta(x,0)\notin \text{int } K \Rightarrow -(f(x)-f(0))\notin \text{int } K$$

and $-g(0)\notin \text{int } Q, \Rightarrow -B\eta(x,0)\in Q$, for all $x\in X_0$, $B\in\partial g(0)$.

However, (f, g) fails to be $(K\times Q)$ generalized type-I at $u=0$ because for $A=\left(-\frac{1}{2},\frac{1}{4}\right)\in\partial f(0)$ and $x=1\in X_0$, we have

$$f(x)-f(0)-A\eta(x,0)=\left(\frac{-1}{2},\frac{-1}{4}\right)\notin K.$$

3 Optimality conditions

Suneja et al. [17] gave the following necessary optimality conditions for (VP).

Theorem 1 [17] (Fritz John type necessary optimality conditions). *Let* f *be* K*-generalized invex and* g *be* Q*-generalized invex at* $x_0\in X_0$ *with respect to same* $\eta: R^n\times R^n\to R^n$. *If* (VP) *attains a weak minimum at* x_0 *then there exist* $\lambda\in K^+$, $\mu\in Q^+$ *not both zero such that*

$$0\in\partial(\lambda f)(x_0)+\partial(\mu g)(x_0) \tag{3}$$

and

$$\mu g(x_0)=0 \tag{4}$$

Definition 13 The problem (VP) is said to satisfy generalized Slater constraint qualification, if there exists $\bar{x}\in R^n$ such that $-g(\bar{x})\in \text{int } Q$.

Theorem 2 [17] (Kuhn-Tucker type necessary optimality conditions). *Let* f *be* K*-generalized invex and* g *be* Q*-generalized invex at* $x_0\in X_0$ *with respect to same* $\eta: R^n\times R^n\to R^n$. *Suppose that generalized Slater constraint qualification is satisfied. If (VP) attains a weak minimum at* x_0*, then there exist* $0\neq\lambda\in K^+$, $\mu\in Q^+$ *such that* (3) *and* (4) *hold.*

We now obtain sufficient optimality conditions for (VP).

Theorem 3 *Let* (f, g) *be* (K × Q) *generalized type-I at* $x_0 \in X_0$ *with respect to* $\eta: X_0 \times R^n \to R^n$ *and suppose that there exist* $0 \neq \lambda^* \in K^+$, $\mu^* \in Q^+$, $(\lambda^*, \mu^*) \neq 0$ *such that*

$$0 \in \partial\left(\lambda^* f\right)(x_0) + \partial\left(\mu^* g\right)(x_0) \tag{5}$$

$$\mu^* g(x_0) = 0 \tag{6}$$

Then x_0 *is a weak minimum of* (VP).

Proof Let if possible x_0 be not a weak minimum of (VP). Then there exists $x \in X_0$ such that

$$f(x_0) - f(x) \in \text{int } K \tag{7}$$

By virtue of (5), there exist $x^* \in \partial(\lambda^* f)(x_0)$, $y^* \in \partial(\mu^* g)(x_0)$ such that

$$x^* + y^* = 0 \tag{8}$$

Since (f, g) is (K × Q) generalized type-I at $x_0 \in X_0$,
Adding (1) and (7) we get,

$$-A\eta(x, x_0) \in \text{int } K, \quad \text{for all } A \in \partial f(x_0),$$

since $\lambda^* \in K^+$, we have

$$\lambda^* A\eta(x, x_0) < 0, \quad \text{for all } A \in \partial f(x_0).$$

As $x^* \in \partial(\lambda^* f(x_0)) = \lambda^* \partial f(x_0)$, we get

$$x^* \eta(x, x_0) < 0.$$

Now using (8) we have $-y^* \eta(x, x_0) < 0$.
$y^* \in \partial(\mu^* g)(x_0) = \mu^* \partial g(x_0)$, we get $y^* = \mu^* B^*$, for some

$$B^* \in \partial g(x_0).$$

Thus,

$$-\mu^* B^* \eta(x, x_0) < 0, B^* \in \partial g(x_0). \tag{9}$$

Now $\mu^* \in Q^+$, therefore from (2) we have, $-\mu^*(g(x_0)) - \mu^* B\eta(x, x_0) \geq 0$, for all $B \in \partial g(x_0)$, which on using (6) gives $-\mu^* B\eta(x, x_0) \geq 0$, for all $B \in \partial g(x_0)$.
This is a contradiction to (9). Hence x_0 is a weak minimum of (VP). □

Theorem 4 *Let* (f, g) *be* (K × Q) *generalized pseudo quasi type-I at* $x_0 \in X_0$ *with respect to* $\eta: X_0 \times R^n \to R^n$ *and suppose there exist* $\lambda^* \in K^+$, $\mu^* \in Q^+$, $(\lambda^*, \mu^*) \neq 0$ *such that (5), (6) hold, then* x_0 *is a weak minimum of (VP).*

Proof Let if possible x_0 be not a weak minimum of (VP) then there exists $x \in X_0$ such that (7) holds.
In view of (5) there exist $x^* \in \partial(\lambda^* f)(x_0)$, $y^* \in \partial(\mu^* g)(x_0)$ such that (8) is satisfied.
Since (f, g) is (K × Q) generalized pseudo quasi type-I at $x_0 \in X_0$, therefore, from (7) we have

$$-A\eta(x, x_0) \in \text{int } K, \quad \text{for all } A \in \partial f(x_0).$$

Now, $\lambda^* \in K^+$, gives

$$\lambda^* A\eta(x, x_0) < 0, \quad \text{for all } A \in \partial f(x_0)$$

which implies,

$$x^*\eta(x, x_0) < 0 \quad \text{as } x^* \in \partial\left(\lambda^* f\right)(x_0). \tag{10}$$

From (6), $\mu^* g(x_0) = 0$, which gives $-g(x_0) \notin \text{int } Q$

$\Rightarrow$ $-B\eta(x, x_0) \in Q$, for all $B \in \partial g(x_0)$ as (f, g) is $(K \times Q)$ generalized pseudo quasi type-I at x_0.

$\Rightarrow$ $\mu^* B\eta(x, x_0) \leq 0$ as $\mu^* \in Q^+$

$\Rightarrow$ $y^*\eta(x, x_0) \leq 0$, as $y^* \in \partial(\mu^* g)(x_0)$

which on using (8) gives $x^*\eta(x, x_0) \geq 0$.

This contradicts (10), hence x_0 is a weak minimum of (VP). □

Theorem 5 *Let* (f, g) *be* $(K \times Q)$ *generalized type-I at* $x_0 \in X_0$ *with respect to* $\eta : X_0 \times R^n \to R^n$ *Suppose there exist* $\lambda^* \in K^{S+}$, $\mu^* \in Q^+$, $(\lambda^*, \mu^*) \neq 0$, *such that* (5) *and* (6) *hold. Then* x_0 *is a minimum of (VP).*

Proof Let if possible, x_0 be not a minimum of (VP), then there exists $x \in X_0$ such that

$$f(x_0) - f(x) \in K/\{0\} \tag{11}$$

As (5) holds, there exist $x^* \in \partial(\lambda^* f)(x_0)$, $y^* \in \partial(\mu^* g)(x_0)$ such that (8) holds.

Since (f, g) is $(K \times Q)$-generalized type-I at $x_0 \in X_0$, therefore proceeding on the similar lines as in proof of Theorem 3 and using (11) we have

$$-A\eta(x, x_0) \in K.$$

As $\lambda^* \in K^{S+}$, $\lambda^* > 0$, we have
$\lambda^* A\eta(x, x_0) < 0$, for all $A \in \partial f(x_0)$.

This leads to a contradiction as in Theorem 3. Hence x_0 is a minimum of (VP). □

Theorem 6 *Let* (f, g) *be strictly* $(K \times Q)$ *generalized pseudo quasi type-I at* $x_0 \in X_0$ *with respect to* $\eta : X_0 \times R^n \to R^n$. *Suppose that there exist* $\lambda^* \in K^+$, $\mu^* \in Q^+$, $(\lambda^*, \mu^*) \neq 0$ *such that* (5) *and* (6) *hold. Then* x_0 *is a minimum of (VP).*

Proof It follows from (5) that there exist $x^* \in \partial(\lambda^* f)(x_0)$ $y^* \in \partial(\mu^* g)(x_0)$ such that (8) holds. Let if possible x_0 be not a minimum of (VP) then there exists $x \in X_0$ such that (11) holds.

Since (f, g) is strictly $(K \times Q)$ generalized pseudo quasi type-I, therefore, we have

$$-A\eta(x, x_0) \in \text{int } K, \text{ for all } A \in \partial f(x_0).$$

Proceeding on the same lines as in proof of Theorem 4, we arrive at a contradiction. Hence x_0 is a minimum of (VP).

Proceeding on the lines of the above theorem the following result can be proved. □

Theorem 7 *Let* (f, g) *be* $(K \times Q)$*-generalized type I at* $x_0 \in X_0$ *with respect to* $\eta : X_0 \times R^n \to R^n$ *and suppose that there exist* $\lambda^* \in K^+$, $\mu^* \in Q^+$, $(\lambda^*, \mu^*) \neq 0$ *such that* (5) *and* (6) *hold then* x_0 *is a minimum solution of the scalarized problem*

$$\begin{aligned}&(VP_{\lambda^*})\ minimize\ (\lambda^* f)(x)\\ &subject\ to - g(x) \in Q\\ &\lambda^* \in K^{s+}\end{aligned}$$

and is a Benson proper minimizer of (VP).

Theorem 8 *Let* (f, g) *be strongly* (K × Q) *generalized pseudo quasi type-I at* $x_0 \in X_0$ *with respect to* $\eta : X_0 \times R^n \to R^n$ *and suppose that there exist* $\lambda^* \in K^+$, $\mu^* \in Q^+$, $(\lambda^*, \mu^*) \neq 0$ *such that* (5) *and* (6) *hold. Then* x_0 *is a strong minimum of (VP).*

Proof By virtue of (5), there exist $x^* \in \partial(\lambda^* f)(x_0)$, $y^* \in \partial(\mu^* g)(x_0)$ such that (8) holds.

Let if possible x_0 be not a strong minimum of (VP), then there exists $x \in X_0$ such that $f(x) - f(x_0) \notin K$.

(f, g) being strongly (K × Q)-generalized pseudo quasi type-I at x_0, we have

$$-A\eta\,(x, x_0) \in \text{int}\ K.$$

Now as in Theorem 4 we arrive at a contradiction, thus proving that x_0 is a strong minimum of (VP). □

4 Duality

We now consider the Mond-Weir type dual of the vector optimization problem (VP).

(VD) K-maximize f(u)

subject to

$$\begin{aligned} &0 \in \partial\,(\lambda f)\,(u) + \partial\,(\mu g)\,(u) \\ &\mu g\,(u) \geq 0 \\ &0 \neq \lambda \in K^+, \mu \in Q^+ \end{aligned} \tag{12}$$

Theorem 9 *Weak duality. Let* x *be feasible for (VP) and (u, λ, μ) feasible for* (VD). *Let* (f, g) *be* (K × Q) *generalized type-I at u with respect to* $\eta : X_0 \times R^n \to R^n$ *and* $(\lambda, \mu) \neq 0$, *then* $f(u) - f(x) \notin \text{int}\ K$.

Proof Since (u, λ, μ) is feasible for (VD) therefore by (12) there exist $x^* \in \partial(\lambda f)(u)$, $y^* \in \partial(\mu g)(u)$ such that

$$x^* + y^* = 0. \tag{13}$$

$$\text{Let if possible } f(u) - f(x) \in \text{int}\ K \tag{14}$$

□

Now (f, g) being (K × Q) generalized type-I at u we have

Adding (1) and (14) we have

$$-A\eta\,(x, u) \in K, \quad \text{for all } A \in \partial f\,(u)\,.$$

As $\lambda \in K^+$, we get

$$\lambda A\eta(x, u) < 0$$

which implies

$$x^*\eta\,(x, u) < 0, \quad \text{as } x^* \in \partial\,(\lambda f)\,(u)\,.$$

By (13) we have $-y^*\eta(x, u) < 0$.

Since $y^* \in \partial(\mu g)(u)$, we obtain $y^* = \mu B^*$, $B^* \in \partial g(u)$.

Thus we have

$$-\,\mu B^*\eta(x, u) < 0. \tag{15}$$

From (2) we get

$$-\mu g(u) - \mu B^{*}\eta(x, u) \geq 0, \text{ as } \mu \in Q^{+}$$

Since u is feasible for (VD), we get

$$-\mu B^{*}\eta(x, u) \geq 0,$$

which contradicts (15). Hence $f(u) - f(x) \notin \text{int } K$.

Theorem 10 Weak duality *Let* $x \in X_0$ *and* (u, λ, μ) *be feasible for* (VD), (f, g) *be* $(K \times Q)$ *generalized pseudo quasi type-I at* u *with respect to* $\eta : X_0 \times R^n \to R^n$ *and* $(\lambda, \mu) \neq 0$ *then* $f(u) - f(x) \notin$ *int K.*

Proof The theorem can be proved on the lines of sufficient optimality Theorem 4. □

Theorem 11 Strong duality *Suppose that* (VP) *attains a weak minimum at* x_0 *and generalized Slater constraint qualification is satisfied. Let* (f, g) *be* $(K \times Q)$ *generalized invex at* x_0 *with respect to* $\eta : R^n \times R^n \to R^n$, *then there exist* $0 \neq \lambda_0 \in K$, $\mu_0 \in Q^{+}$ *such that* (x_0, λ_0, μ_0) *is a feasible solution of* (VD). *Further if the conditions of Weak Duality Theorem 9 hold for all* $x \in X_0$ *and feasible* (u, λ, μ) *of (VD), then* (x_0, λ_0, μ_0) *is a weak maximum of (VD) and the values of objective functions for (VP) and (VD) are equal.*

Proof As x_0 is a weak minimum of (VP), by Theorem 2, there exist $0 \neq \lambda_0 \in K^{+}$, $\mu_0 \in Q^{+}$ such that

$$0 \in \partial(\lambda_0 f)(x_0) + \partial(\mu_0 g)(x_0)$$

and

$$\lambda_0 g(x_0) = 0,$$

which clearly shows that (x_0, λ_0, μ_0) is a feasible solution of (VD) and the values of two objectives functions are equal. Further if (x_0, λ_0, μ_0) is not a weak maximum of (VD), then there exists a feasible solution (u, λ, μ) of (VD) such that

$$f(u) - f(x_0) \in \text{int } K,$$

which contradicts Theorem 9. Hence (x_0, λ_0, μ_0) is a weak maximum of (VD). □

Acknowledgments The authors are grateful to the anonymous referees for their useful comments and suggestions which have improved the presentation of the paper.

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